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Maps of Intervals with Indifferent Fixed Points: Thermodynamic Formalism and Phase Transitions

Thomas Prellberg[1] **and Joseph Slawny**[1,2]

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We develop the thermodynamic formalism for a large class of maps of the interval with indifferent fixed points. For such systems the formalism yields one-dimensional systems with many-body infinite-range interactions for which the thermodynamics is well defined but Gibbs states are not. (Piecewise linear systems of this kind yield the soluble, in a sense, Fisher models.) We prove that such systems exhibit phase transitions, the order of which depends on the behavior at the indifferent fixed points. We obtain the critical exponent describing the singularity of the pressure and analyze the decay of correlations of the equilibrium states at all temperatures. Our technique relies on establishing and exploiting a relation between the transfer operators of the original map and its suitable (expanding) induced version. The technique allows one also to obtain a version of the Bowen–Ruelle formula for the Hausdorff dimension of repellers for maps with indifferent fixed points, and to generalize Fisher results to some nonsoluble models.

KEY WORDS: Nonhyperbolic maps; thermodynamic formalism; phase transitions; transfer operators; inducing.

The thermodynamic formalism[25,28] proved to be a powerful tool in the ergodic theory of hyperbolic and, in particular, expanding maps.[26] A central role is here played by the transfer (or Ruelle–Perron–Frobenius) operator. The fact that the map is expanding allows one to express thermodynamic and statistical characteristics of the system (free energy, equilibrium states,...) in terms of the transfer operator, and results in regularity

[1] Department of Nuclear Physics, Weizmann Institute of Science, 76100 Rehovot, Israel.

[2] Meyerhoff Visiting Professor, on leave from the Center for Transport Theory and Mathematical Physics, Virginia Polytechnic Institute and State University, Blacksburg, Virginia 24061.

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properties of both. In particular, one obtains a statistical mechanics system with a fast-decaying interaction and, correspondingly, a transfer operator with compactness properties, which allows for quite a complete analysis of such systems. As a consequence, one has fast convergence to the thermodynamic limit and smoothness of thermodynamic functions (no phase transitions).

These regularity properties disappear when one passes to nonhyperbolic maps, as has been demonstrated convincingly in recent works, mostly by theoretical physicists. Numerical analysis and calculations in some soluble models exhibit both singularities and slow convergence to the thermodynamic limit.[8,9,13] Some insights have been gained into the origin of the singularities, in particular by relating the phase transitions to that of Fisher models[20,29,32,33]; see the closing remarks here. We note, however, that in natural approximations of the systems by Fisher models the discarded parts of the interactions are not small in any obvious sense, having, in particular, infinite "energy norm."

On the other hand, in a number of mathematical works, the method of *inducing*, and its variants, has been used to investigate absolutely continuous invariant measures for nonhyperbolic maps of the interval.[1,5,18,23]

Apart from a remark of Walters[31] on a relation between pressures of a soluble system and its induced version, we are aware of no work relating thermodynamics and transfer operators of a system and its induced version. The aim of this paper is to establish such a relation and to show that it yields quite a complete version of the thermodynamic formalism for almost expanding maps (defined below), with a good insight into the nature of singularities in such systems; this relation can be considered as a version of the renormalization group idea. The same inducing method allows us to obtain results on singularities in the spectrum of the transfer operator for some unimodal maps, confirming, in particular, results and conjectures of refs. 4 and 20. However, in the unimodal case the thermodynamic significance of the results is yet to be clarified.

To simplify the exposition, we restrict our attention to *almost expanding* maps of the interval $I = [0, 1]$, which are defined below. Later, we indicate what changes have to be made to treat more general almost expanding maps and some other nonhyperbolic systems.

We consider a piecewise monotone transformation f of the interval $I = [0, 1]$; there exists a finite partition of I into intervals $I_0, I_1, ..., I_K$, such that for each interval I_i, f extends to a function f_i on its closure $\bar{I}_i$ with Hölder-continuous derivative f_i'. We denote by F_i the map inverse to f_i.

f is *almost expanding* if $|f'|$ is larger than 1 in the interior of each I_i (it may be equal to 1 at the endpoints of the intervals).

In order to keep the formulas simple, we restrict ourselves to a partition into two intervals $I_0 = [0, a[$ and $I_1 = [a, 1]$ and we suppose that f_0 and f_1 are onto I. Furthermore, we now assume that f has an indifferent fixed point at 0, and that $|f'|$ is larger than some $\lambda_0 > 1$ on $f^{-1}(I_1)$. A typical f in this class is the *Farey map*,[11] where $a = 1/2$, $f_0 = x/(1-x)$, and $f_1 = (1-x)/x$ (see Fig. 1).

For real β, the *transfer operator* $\mathscr{L}_\beta$ associated with the transformation f is defined by

$$\mathscr{L}_\beta \Phi(x) = \sum_{y: fy = x} \frac{\Phi(y)}{|f'(y)|^\beta} \tag{1}$$

with the natural convention at the endpoints of the interval. $\mathscr{L}_\beta$ acts on suitable Banach spaces of functions on I.[25]

If f is expanding, then there exists a unique equilibrium state, which is also a Gibbs state, and the pressure $P(\beta \log |f'|)$ is analytic in β. Moreover, the pressure is given by the logarithm of the largest eigenvalue of $\mathscr{L}_\beta$ in the space of continuous functions on I,[31] or in the space of functions with bounded variation on I.[3] For some nonexpanding transformations f inducing will be used to obtain an expanding induced system, and results in the induced system will be related to the original one. This allows us to prove below the existence of phase transitions for almost expanding maps with indifferent fixed points, and analyze corresponding singularities of thermodynamic functions and clustering properties of equilibrium states.

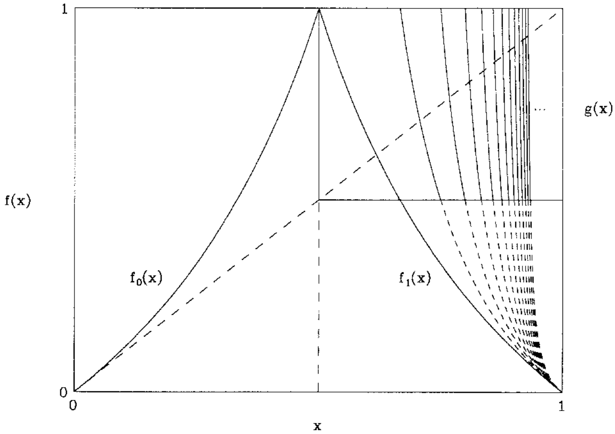

Fig. 1. The Farey map f with the induced map g in the upper right corner. f_0 and f_1 are the branches of f on $\bar{I}_0$ and $\bar{I}_1$. The dashed lines show the extension of branches of the induced map to the whole interval.

For a partition into two intervals, the transfer operator $\mathcal{L}_\beta$ reads

$$\mathcal{L}_\beta \Phi = \mathcal{L}_{0\beta} \Phi + \mathcal{L}_{1\beta} \Phi, \qquad \text{where} \quad \mathcal{L}_{i\beta} \Phi = |F_i'|^\beta \, \Phi \circ F_i = \mathcal{L}_\beta(\chi_i \Phi) \tag{2}$$

and χ_i is the characteristic function of I_i, $i = 0, 1$.

To define an appropriate *induced transformation*,[6] let J be a subinterval of I, and let J_n be the set of points which return to J after exactly n iterations,

$$J_n = \{x \in J \mid fx \notin J, ..., f^{n-1}x \notin J, f^n \in J\} \tag{3}$$

Defining $n(x) = n$ for $x \in J_n$, the *first return* or *induced map* g is defined by

$$gx = f^{n(x)}x \tag{4}$$

For an almost expanding map of the interval, the induced transformation is defined up to a countable set of points. In our case, only the point $x = 1$ does not return to J.

Furthermore, define the *modified* (by the parameter z) transfer operator $\mathcal{M}_{\beta z}$ for the induced map,

$$\mathcal{M}_{\beta z} \Psi(x) = \sum_{y:\, gy = x} \frac{z^{n(y)}}{|g'(y)|^\beta} \Psi(y) \tag{5}$$

Now induce on $J = I_1$. Then the sets J_n are the intervals

$$J_n = F_1 F_0^{n-1}(J) \tag{6}$$

and g maps each J_n monotonically onto J. Denoting by $G_n\colon J \to J_n$ the inverse of $g\,|_{J_n}$, we have

$$G_n = F_1 F_0^{n-1}\,|_J \tag{7}$$

Figure 1 illustrates the inducing for the Farey map. The induced map is shown in the upper right corner.

Due to our assumptions, g is expanding, i.e.,

$$\sup_{n \geq 1} \sup_{x \in J} |G_n'(x)| < 1 \tag{8}$$

The modified transfer operator $\mathcal{M}_{\beta z}$ takes the form

$$\mathcal{M}_{\beta z} \Psi = \sum_{n=1}^{\infty} z^n \, |G_n'|^\beta \, \Psi \circ G_n \tag{9}$$

which for the Farey system is

$$\mathscr{M}_{\beta z}\Psi(x) = \sum_{n=1}^{\infty} \frac{z^n}{(1+nx)^{2\beta}}\,\Psi\left(\frac{1+(n-1)x}{1+nx}\right) \tag{10}$$

We note that because of the bound (8) the series in (9) is pointwise convergent for any bounded Ψ and for $|z|<1$.

We now reinterpret the last expression for $\mathscr{M}_{\beta z}$ as follows. By (7), $F_1F_0^{n-1}$ extends G_n to all of I. Moreover, F_1, and therefore also $F_1F_0^{n-1}$, maps I into J. Hence, defining for Ψ, β, and z as in (9),

$$\mathscr{M}_{\beta z}^{+}\Psi = \sum_{n=1}^{\infty} z^n\,|(F_1F_0^{n-1})'|^{\beta}\,\Psi\circ F_1F_0^{n-1} \tag{11}$$

we obtain that $\mathscr{M}_{\beta z}^{+}\Psi$ extends $\mathscr{M}_{\beta z}\Psi$ to I. Moreover, the action of the operator $\mathscr{M}_{\beta z}^{+}$ is also well defined for functions on I.

To relate $\mathscr{L}_\beta$ and $\mathscr{M}_{\beta z}$, we note that

$$\mathscr{L}_{0\beta}^{n-1}\mathscr{L}_{1\beta}\Psi = |(F_1F_0^{n-1})'|^{\beta}\,\Psi\circ F_1F_0^{n-1} \tag{12}$$

Here the operator product is defined on functions on I, but also holds when restricted to functions on J. By (7) and (12),

$$\mathscr{M}_{\beta z}^{+}\Psi = \sum_{n=1}^{\infty} z^n\mathscr{L}_{0\beta}^{n-1}\mathscr{L}_{1\beta}\Psi \tag{13}$$

which is equal to $(1-z\mathscr{L}_{0\beta})^{-1}z\mathscr{L}_{1\beta}\Psi$ for $|z|$ smaller than the radius of convergence of this power series. This radius is equal to $r(\mathscr{L}_{0\beta})$, the spectral radius of $\mathscr{L}_{0\beta}$. In our case, $r(\mathscr{L}_{0\beta})=1$, due to the fact that f is almost expanding and that $f'(0)=1$. Because $|G_n'(0)|=1$, $\mathscr{M}_{\beta z}^{+}\Psi$ is in general unbounded for $z=1$, exhibiting a singularity at zero.

Now, (13) yields the desired operator relations between $\mathscr{L}_\beta$ and $\mathscr{M}_{\beta z}$. For any function Φ on I

$$(1-z\mathscr{L}_{0\beta})(1-\mathscr{M}_{\beta z}^{+})\Phi = (1-z\mathscr{L}_\beta)\Phi \tag{14}$$

and, for any function Ψ on J, one has

$$(1-z\mathscr{L}_\beta)\,\mathscr{M}_{\beta z}^{+}\Psi = z\mathscr{L}_{1\beta}(1-\mathscr{M}_{\beta z})\Psi \tag{15}$$

We now discuss how in suitable Banach spaces of functions these identities relate the spectrum of $\mathscr{L}_\beta$ outside the disk of radius $r(\mathscr{L}_{0\beta})$ to the spectrum of $\mathscr{M}_{\beta z}$; the spectrum of $\mathscr{L}_\beta$ inside this disk is considered later.

Let Φ be an eigenfunction of $\mathscr{L}_\beta$ with eigenvalue $z^{-1}\geqslant 1$. Then Φ is an eigenfunction of $\mathscr{M}_{\beta z}^{+}$ with eigenvalue 1, and hence, the restriction Ψ of Φ

to J is an eigenfunction of $\mathcal{M}_{\beta z}$ with eigenvalue 1. Also, if z^{-1} is in the resolvent set of $\mathcal{L}_\beta$, then 1 is in the resolvent set of $\mathcal{M}^+_{\beta z}$ and, hence, of $\mathcal{M}_{\beta z}$.

Now, let Ψ be an eigenfunction of $\mathcal{M}_{\beta_z}$ with eigenvalue 1. Then the extension $\Phi = \mathcal{M}^+_{\beta_z} \Psi$ of Ψ to I is an eigenfunction of $\mathcal{L}_\beta$ with eigenvalue z^{-1}, provided that the extension Φ is in the domain of $\mathcal{L}_\beta$. Also, if 1 is in the resolvent set of $\mathcal{M}_{\beta_z}$ then z^{-1} is in the resolvent set of $\mathcal{L}_\beta$.

The operator $\mathcal{M}_{\beta z}$ is well defined for $z \in]0, 1]$ and $\beta > 1/2$, and has the following positivity and monotonicity properties. For bounded Φ, Ψ we write $\Phi \leqslant \Psi$ if $\Phi(x) \leqslant \Psi(x)$ for any $x \in J$. Then for any $\Psi \geqslant 0$

$$\mathcal{M}_{\beta z} \Psi \geqslant 0 \tag{16}$$

and

$$\mathcal{M}_{\beta z} \Psi \leqslant \lambda_0^{\beta' - \beta} \mathcal{M}_{\beta' z} \Psi, \qquad \mathcal{M}_{\beta z} \Psi \leqslant \frac{z}{z'} \mathcal{M}_{\beta' z} \tag{17}$$

for $\beta \geqslant \beta'$ and $z \leqslant z'$; here λ_0 is the lhs of the inequality (8).

We will discuss now the spectral properties of $\mathcal{M}_{\beta z}$ and the existence of phase transitions in the particularly simple case of a piecewise analytic f; following refs. 24 and 22, we let the operators act on the Banach space of functions analytic in a suitable (complex) neighborhood of J. The induced map g is expanding, and therefore $\mathcal{M}_{\beta z}$ maps functions analytic in a neighborhood of J into functions that are analytic in a larger neighborhood. Due to this property, $\mathcal{M}_{\beta z}$ is a compact (in fact trace-class) operator.

Now consider $1 \geqslant z > 0$. Then the operator $\mathcal{M}_{\beta z}$ is not only positive but also 1-bounded, i.e., for a nonzero $\Psi \geqslant 0$ there exist n, α_1, and α_2 such that

$$\alpha_1 1 < (\mathcal{M}_{\beta z})^n \Psi < \alpha_2 1 \tag{18}$$

(the upper bound is trivial). Thus, there exists a unique (up to a factor) positive eigenvector $\Psi_{\beta z}$ of $\mathcal{M}_{\beta z}$, the corresponding eigenvalue $\lambda_{\max}(\mathcal{M}_{\beta z})$ is simple, and all other eigenvalues are strictly smaller in modulus.[21,22] This yields also a spectral gap for $\mathcal{M}_{\beta z}$.

Denoting by Λ the Lebesque measure, one has

$$\Lambda(\mathcal{M}_{1,1} \Psi) = \Lambda(\Psi) \tag{19}$$

for any Ψ; this is standard in the theory of the transfer operator. Combining this with the inequalities (17) and the fact that, obviously, $\lambda_{\max}(\mathcal{M}_{\beta z}) \to 0$ as $z \to 0$, one obtains that for any $\beta < 1$ there is unique $z(\beta) < 1$ such that

$$\lambda_{\max}(\mathcal{M}_{\beta z(\beta)}) = 1 \tag{20}$$

and that $z(\beta)$ is a strictly increasing function of β with $z(\beta) \nearrow 1$ as $\beta \nearrow 1$. Moreover, by standard perturbation theory of simple eigenvalues (ref. 19, Chapter VII, Theorem 1.9), $\beta \mapsto z(\beta)$ is analytic.

Since, as we show below,[3] $P(\beta) = -\log z(\beta)$ for $\beta < 1$, this argument accounts for the graph of Fig. 2 on the interval [0, 1]. Furthermore, since $P(\beta) \geqslant 0$ and $P(\beta)$ is a decreasing function of β, $P(\beta) = 0$ for $\beta \geqslant 1$, and therefore $\beta = 1$ is a point of a phase transition.

To identify $P(\beta)$ with $-\log z(\beta)$ for $\beta \in [0, 1]$, we note first that, by (18),

$$\inf_{x \in J} \Psi_{\beta z(\beta)}(x) > 0$$

Therefore $\Phi_\beta := \mathscr{M}^+_{\beta z(\beta)} \Psi_{\beta z(\beta)}$ is a strictly positive continuous function on I, which by (15) is an eigenfunction of $\mathscr{L}_\beta$ with eigenvalue $1/z(\beta)$. Moreover, applying Theorem 2.1 of ref. 30 to the normalized version $\bar{\mathscr{L}}$ of $\mathscr{L}_\beta$,

$$\bar{\mathscr{L}}\Phi(x) = z(\beta)\, \Phi_\beta(x)^{-1}\, (\mathscr{L}_\beta(\Phi_\beta \Phi))(x)$$

we obtain that

$$P(\beta) = -\log z(\beta) \tag{21}$$

This concludes our analysis of the piecewise analytic case.

[3] Here and in what follows we write $P(\beta)$ in place of $P(\beta \log |f'|)$.

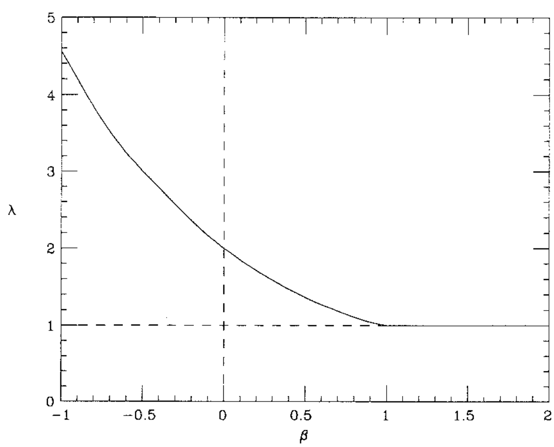

Fig. 2. The β dependence of the largest eigenvalue λ of the transfer operator.

We turn now to the more general case of maps f for which f' is merely Hölder continuous, and indicate how inducing and the recent results of refs. 3 and 27 yield additional information on the spectrum of $\mathscr{L}_\beta$ and the clustering properties of equilibrium states.

We let the operators $\mathscr{L}_\beta$ and $\mathscr{M}_{\beta z}$ act on the Banach spaces $BV(I)$ and $BV(J)$ of functions of bounded variation on the intervals I and J, respectively. Then, using the fact that g is expanding, we obtain that for $z \in]0, 1[$ and $\beta < 1$ the spectrum of $\mathscr{M}_{\beta z}$ is as indicated on Fig. 3a: $\mathscr{M}_{\beta z}$ is quasicompact, its spectral radius is a simple eigenvalue $\lambda_{\max}(\mathscr{M}_{\beta z})$ with a positive Hölder continuous eigenfunction $\Psi_{\beta z}$, the essential spectral radius of $\mathscr{M}_{\beta z}$, $r_{\text{ess}}(\mathscr{M}_{\beta z})$, is strictly smaller than $\lambda_{\max}(\mathscr{M}_{\beta z})$, and the rest of the spectrum of $\mathscr{M}_{\beta z}$ consists of isolated eigenvalues of a finite multiplicity of modulus strictly smaller than $\lambda_{\max}(\mathscr{M}_{\beta z})$. Furthermore, using the inequalities (17), one shows as before that there is a unique solution $z(\beta)$ to (20), and that $z(\beta)$ has the monotonicity and analyticity properties stated after (20).

Using now (14) and (15), we show that $\mathscr{L}_\beta$ acting on $BV(I)$ is also quasicompact, that $r_{\text{ess}}(\mathscr{L}_\beta) = 1$, that $z(\beta)^{-1}$ is a simple eigenvalue of $\mathscr{L}_\beta$, equal to its spectral radius, with the corresponding eigenfunction $\mathscr{M}^{+}_{\beta z(\beta)} \Psi_{\beta z(\beta)}$ strictly positive and Hölder-continuous, and that, as in the case of $\mathscr{M}_{\beta z}$, the rest of the spectrum of $\mathscr{L}_\beta$ consists of isolated eigenvalues of a finite multiplicity of modulus strictly smaller than $z(\beta)^{-1}$ (Fig. 3b). In particular, we obtain that $\mathscr{L}_\beta$ has a spectral gap whenever $\beta < 1$, and since $z(\beta) \nearrow 1$ as $\beta \nearrow 1$, this spectral gap goes to zero as $\beta \nearrow 1$. Also, since $P(\beta)$ is equal to the spectral radius of $\mathscr{L}_\beta$ in general,[3] we obtain again that (21) holds, extending to the present context results well known for expanding maps.

We discuss now the behavior at the critical point $z = 1$. This depends

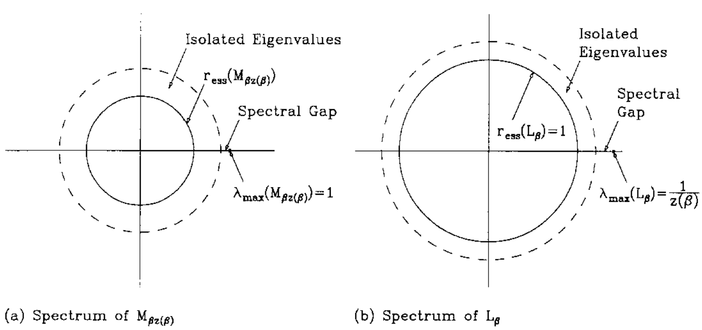

Fig. 3. The structure and the connection between the spectra of (a) the modified transfer operator $\mathscr{M}_{\beta z(\beta)}$ and (b) the transfer operator $\mathscr{L}_\beta$, for $\beta < 1$.

on the asymptotic form of f near the indifferent fixed point 0. Assume a power law in the form

$$f(x) = x + cx^{\bar{z}}[1 + r(x)] \tag{22}$$

with exponent $\bar{z} > 1$, some constant $c > 0$, with $r(x) = O(x^{\varepsilon})$ as $x \to 0$, for some $\varepsilon > 0$, and with $r'(x)$ monotone in a neighborhood of 0 (for instance, r' analytic at 0).

Then we have the asymptotic expression

$$F_0^n(x) = g^n(x)[1 + O(g^n(x))^{\varepsilon}]$$

uniformly in n for $x \to 0$, where

$$g(x) = (x^{-(\bar{z}-1)} + a)^{-1/(\bar{z}-1)}$$

is the fixed point of the renormalization transformation

$$T_{\alpha}\, g(x) = \alpha g^2(x/\alpha), \qquad \alpha = 2^{1/(\bar{z}-1)}$$

for the intermittency boundary condition[17]

$$g(0) = 0, \qquad g'(0) = 1$$

(The exponent $\bar{z}$ must not be confused with the parameter $z = \lambda^{-1}$ used earlier.)

From this we obtain the asymptotic expansion near the phase transition point:

1. The system exhibits a first-order phase transition for $1 < \bar{z} < 2$:

$$P(\beta) = \text{const} \cdot (1 - \beta) + o(1 - \beta) \tag{23}$$

2. For $\bar{z} = 2$, the case of the Farey map,

$$-P(\beta) \log P(\beta) = \text{const} \cdot (1 - \beta) + o(1 - \beta) \tag{24}$$

3. For $\bar{z} > 2$

$$P(\beta) = \text{const} \cdot (1 - \beta)^{\bar{z}-1} + o((1 - \beta)^{\bar{z}-1}) \tag{25}$$

Our results agree with those of refs. 11, 29, and 32.

We turn now to the statistical mechanics of the system, i.e., to a description of its equilibrium states.[25] We again use inducing in combination with the variational principle and standard results of the thermodynamic formalism.[25,30] We obtain among other results the description of equilibrium states and their clustering properties:

For $\beta < 1$ there is a unique equilibrium state. Though not a Gibbs state, this state has exponential decay of correlations, with the correlation length diverging as β approaches 1. For $\beta > 1$ there is again a unique, and β-independent, equilibrium state δ_0 concentrated at the point $\{0\}$.

The situation at $\beta = 1$ and the order of the phase transition depend on the exponent $\bar{z}$:

1. For $\bar{z} < 2$ one has two extremal equilibrium states, one, ρ, obtained as a limit as $\beta \nearrow 1$ of the unique equilibrium states; the other is the δ_0 mentioned above. We also obtain that although the phase transition is of a first order, ρ has polynomially decaying correlations.

2. For $\bar{z} \geqslant 2$ one has unique equilibrium state, namely δ_0. The order of the phase transition is as described above. By the general thermodynamic formalism, the limit as $\beta \nearrow 1$ of the unique equilibrium state is equal to δ_0.

In addition, there is a unique, up to a factor, invariant measure which is absolutely continuous with respect to the Lebesque measure; however, unlike in the case of $\bar{z} < 2$, this measure is infinite.

We mention now further extensions of the formalism and some results it yields. Some of these will be treated in a longer paper in preparation.

1. One can define the modified transfer operator (5) in a much more general setting, when I and J are not necessarily intervals, or subsets of the real line. Abusing somewhat the notation. $\mathcal{M}_{\beta z}$ can be written in terms of $\mathcal{L}_\beta$ and the sets J_n of (3):

$$\mathcal{M}_{\beta z} \Psi = \sum_{n=1}^{\infty} z^n \mathcal{L}_\beta^n(\chi_{J_n} \Psi)$$

where χ_{J_n} is the characteristic function of J_n. Defining

$$\mathcal{L}_{0\beta} \Phi = \mathcal{L}_\beta(\chi_{J^c} \Phi) \qquad \text{and} \qquad \mathcal{L}_{1\beta} \Phi = \mathcal{L}_\beta(\chi_J \Phi)$$

we again obtain that $\mathcal{M}^+_{\beta z}$ given by (13) extends $\mathcal{M}_{\beta z}$ and that the relations (14) and (15) hold.

2. The present results extend, almost verbatim, to a situation when f has more than two branches; a slight variation of the formalism works for periodic indifferent points.

3. Most of the results depend on the expanding nature of the induced map g, not of f.

4. A natural modification of the formalism extends it to the case of one-dimensional repellers, or, "cookie cutters,"[4] with indifferent fixed points. In particular, one obtains a version of the Bowen–Ruelle formula, which now says that the Hausdorff dimension of the corresponding Julia set is given by β at which the phase transition occurs. (In the situation of the present paper this β_{cr} is 1, corresponding to the fact that the Julia set of consists of all the interval.) When these results were described at seminars at IHES and the Tel Aviv University, we were informed by E. Cowley and Jon Aaronson about similar results (on Hausdorff dimension, not on the transfer operators) of ref. 10 for some Julia sets.

5. The Fisher models[14] are soluble in the sense that their induced versions have only one-point interactions. Our formulation allows one to generalize these results to a large class of models that are no longer soluble in this sense, with induced versions having interaction of an infinite range, as is the case with statistical mechanics systems arising from smooth maps of the interval. The modified transfer operator $\mathcal{M}_{\beta z}$ can be considered as an operator version of the grand canonical ensemble. For Fisher models, $\mathcal{M}_{\beta z}$ reduces to multiplication by the grand partition function of a single cluster as in ref. 32.

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